Quantumness of noisy quantum walks: a comparison between measurement-induced disturbance and quantum discord

Balaji R. Rao and R. Srikanth*

Poornaprajna Institute of Scientific Research, Sadashivnagar, Bengaluru- 560 080.

C. M. Chandrashekar†

Center for Quantum Sciences, The Institute of Mathematical Sciences, Chennai 600113, India.

Subhashish Banerjee

Indian Institute of Technology Rajasthan, Jodhpur 342011, India.

Noisy quantum walks are studied from the perspective of comparing their quantumness as defined by two popular measures, measurement-induced disturbance (MID) and quantum discord (QD). While the former has an operational definition, unlike the latter, it also tends to overestimate non-classicality because of a lack of optimization over local measurements. Applied to quantum walks, we find that MID, while acting as a loose upper bound on QD, still tends to reflect correctly the trends in the behavior of the latter. However, there are regimes where its behavior is not indicative of non-classicality: in particular, we find an instance where MID increases with the application of noise, where we expect a reduction of quantumness.

I. INTRODUCTION

Many useful quantities in quantum information theory (such as various quantifications of entanglement and channel capacities) lack an operational definition. Quantifying the degree of nonclassicality or quantumness in a state is one such. Intuitively, we expect that entanglement captures all of the nonclassicality in a correlation. We now know that in general this is not the case and that, for mixed states, non-classicality, nonlocality, and entanglement are not identical.

In the case of quantifying the quantumness of a bipartite state, following the proposal of quantum discord (QD) [1], which requires an extremization over local measurement strategies, measurement-induced disturbance (MID) [2] was proposed as an operational measure. Recently, QD has received several operational interpretations, in terms of the efficiency of Maxwell's demon [3], the entanglement consumed [4] or quantum communication [5] during state merging, and distillable entanglement in quantum measurement [6]. However, the difficulty posed by the required optimization remains. In contrast, MID requires no such optimization, instead it uses the local measurement strategy defined by the diagonalization of the reduced density operators. If MID were a good indicator of non-classicality, in particular, if it correctly reflected the behavior of QD, we would have a happy instance of an operational proxy for genuine non-classicality. However, Ref. [7] has reported several difficulties with the use of MID for a two-qubit system. In particular, there are states of vanishing (symmetrized) discord for which MID is maximal. One way to amelio-

rate the performance of MID is to optimize it over all possible local measurements [7]. In this work, we compare these two indicators of non-classicality by applying them to unitary and noisy discrete-time quantum walk (DTQW), treated as a $(2 \times k)$-dimensional system.

Quantum walks (QWs) [8, 9], which are the quantum analogs of classical random walks (CRWs), have been extensively studied as a quantum algorithm [10–16], to demonstrate coherent control over atoms [17], to explain phenomena such as the breakdown of an electric-field driven system [18], and as direct experimental evidence for wavelike energy transfer within photosynthetic systems [19, 20]. Decoherence in a QW and the transition of a QW to a CRW is quite important from the viewpoint of practical implementation, and it has been studied by various authors [21–27]. In particular, in Refs. [24–26], we investigated some qualitatively different ways in which environmental effects suppress quantum superposition in a QW on a line and on an $n-$cycle.

This report is arranged as follows. In Sec. II we briefly introduce the DTQW model on a line and on an $n-$cycle as well as the noise channel used for our study. In Sec. III we compare and contrast the quantumness of a QW subjected to noise, as computed by QD and MID, to quantify the quantumness and investigate the proximity of the outcomes using the two measures, MID and QD. Finally, we conclude in Sec. IV.

II. DISCRETE-TIME QUANTUM WALK ON A LINE AND AN $n-$CYCLE

A DTQW in one dimension is modeled as a particle consisting of a two-level coin (a qubit) existing in the Hilbert space $\mathcal{H}_c$, spanned by $|0\rangle$ and $|1\rangle$, and a position degree of freedom existing in the Hilbert space $\mathcal{H}_p$, spanned by $|\psi_x\rangle$, where $x \in \mathbb{I}$, the set of integers. In an n-
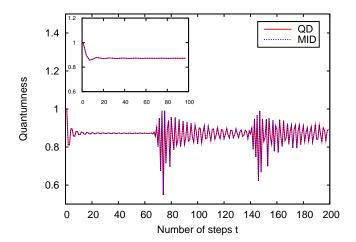
FIG. 1: (Color online) QD and MID for a unitary walk using $B_{\pi/2}$ as a quantum coin operation on a 51-cycle (inset is for a 100-iteration walk on a line). For a noiseless walk, the quantumness using MID and QD is the same (see Theorem 1).

cycle walk, there are only n allowed positions, and in addition the periodic boundary condition $|\psi_x\rangle = |\psi_{x \bmod n}\rangle$ is imposed. For our study, a t-step coined QW is generated by iteratively applying a unitary operation W, which acts on the Hilbert space $\mathcal{H}_c \otimes \mathcal{H}_p$:

$$|\Psi_t\rangle = W^t|\Psi_{in}\rangle, \qquad (2.1)$$

where $|\Psi_{in}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) \otimes |\psi_0\rangle$ is an initial state of the particle and $W \equiv U(B \otimes \mathbb{1})$, where $U(2) \ni B = B_\theta \equiv \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix}$ is the coin operation. U is the controlled-shift operation

$$U \equiv |0\rangle\langle 0| \otimes \sum_x |\psi_x - 1\rangle\langle\psi_x| + |1\rangle\langle 1| \otimes \sum_x |\psi_x + 1\rangle\langle\psi_x|. \qquad (2.2)$$

For an n-cycle, $|\psi_x - 1\rangle$ and $|\psi_x + 1\rangle$ are replaced by $|\psi_{x-1 \bmod n}\rangle$ and $|\psi_{x+1 \bmod n}\rangle$, respectively. The probability of finding the particle at site x after t steps is given by $p(x,t) = \langle\psi_x|\mathrm{Tr}_c(|\Psi_t\rangle\langle\Psi_t|)|\psi_x\rangle$.

To quantify quantumness when noise is applied to a DTQW, we will consider the amplitude-damping channel [28] parametrized by λ which has the following operator-sum representation :

$$E_0 \equiv \begin{bmatrix} \sqrt{1-\lambda} & 0 \\ 0 & 1 \end{bmatrix} \quad ; \quad E_1 \equiv \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{\lambda} \end{bmatrix} \qquad (2.3)$$

where λ ranges from the noiseless case (0) to that of maximum noise (1). More general noise models can be used, such as a dissipative interaction in the presence of a squeezed thermal bath [29], but the above simple model captures all the essential physics, and is hence found to be sufficient for present purposes.
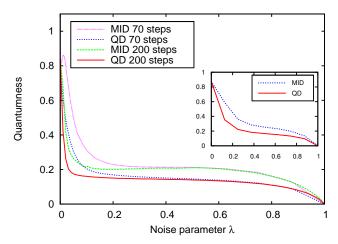

FIG. 2: (Color online) QD and MID for a quantum walk with the increase of noise level due to an amplitude-damping channel on a 51-cycle, after 70- and 200-iterations, respectively (inset is for a walk on a line after 100-iterations). Owing to the closed, periodic dynamics in the n-cycle, the effect of very little noise is amplified leading to a steep reduction in the quantumness. We note that MID and QD follow a similar trend. In general, QD $\leq$ MID [7].

III. QUANTFYING QUANTUMNESS

A number of measures for quantifying quantumness exist [1, 2, 30–32], most of which are not operationally defined. Except in the simplest cases, extensive numerics would be needed. From these, we selected measurement-induced disturbance (MID) [2], which has an operational definition, and quantum discord (QD) [1], which involves extremization over measurement strategies. We consider the classicalization of a QW on a line and on an n-cycle under the influence of the amplitude-damping channel Eq. (2.3).

a.   *Measurement-induced disturbance.* Given a bi-partite state ρ existing in the Hilbert space $\mathcal{H}_C \otimes \mathcal{H}_P$, let the reduced density matrices be denoted by ρ_C and ρ_P. Let $\rho_C = \sum_i p_C^i \Pi_C^i$ and $\rho_P = \sum_j p_P^j \Pi_P^j$. The measurement induced by the spectral resolution of the reduced states is

$$\Pi(\rho) \equiv \sum_{j,k} \Pi_C^j \otimes \Pi_P^k \rho \Pi_C^j \otimes \Pi_P^k, \qquad (3.1)$$

which may be considered classical in the sense that there is a (unique) local measurement strategy, namely, Π, that leaves $\Pi(\rho)$ unchanged. This strategy is special in that it produces a classical state in ρ while keeping the reduced states invariant.

According to Luo [2], a reasonable measure of quantumness is MID, given by

$$Q(\rho) = I(\rho) - I[\Pi(\rho)], \qquad (3.2)$$

where $I(\cdot)$ is mutual information. Accordingly, Eq. (3.2)

is interpreted as the difference between the total and classical correlations.

b. Quantum discord. Quantum discord [1] is given by:

$$\mathcal{D}(P|C) = \mathcal{I}(P:C)^Q - \mathcal{J}(\mathcal{P}:\mathcal{C})^Q_{\{\Pi_j^C\}} \qquad (3.3)$$

$$= S(C) - S(P,C) + S(P|\{N_j^C\}) \qquad (3.4)$$

$$= S(P|\{N_j^C\}) - S(P|C), \qquad (3.5)$$

where $S(P|\{N_j^C\}) = \sum_j p_j S(\rho_{X|N_j^C})$. $\rho_{X|N_j^C} = \mathrm{Tr}_C[\mathbb{I}_P \otimes N_j^C \rho_{P,C}] / \mathrm{Tr}[N_j^C \rho_{P,C}]$ is the state of P after outcome N_j^C. This is in general computationally very intensive. However, it has been shown that for qubit systems it suffices to consider rank-1 positive operator valued measures (POVMs) [34], which for qubits reduces to projective measurements.

We have numerically evaluated $\mathcal{D}(P|C)$ by minimizing Eq. 3.5 by performing projective measurement over all bases for C parametrized by α and β : $\{\cos(\alpha)|0\rangle + e^{i\beta}\sin(\alpha)|1\rangle, e^{-i\beta}\sin(\alpha)|0\rangle - \cos(\alpha)|1\rangle\}$. Because of Theorem 1 below, a comparison of QD and MID is interesting only for mixed states.

Theorem 1 *For pure states, MID, QD and entanglement are identical.*

$$\mathcal{D}(P|C) = S(P|\{N_j^C\}) - S(P|C)$$
$$= S(P|\{N_j^C\}) - S(P,C) + S(C)$$
$$= S(P|\{N_j^C\}) + S(C).$$

Proof. The expression $P|\{N_j^C\}$ is the state of P after C is measured. In the case of entangled pure bipartite states, by virtue of Schmidt decomposition, when the outcome of measuring C is known, the state of P after measuring C is also exactly known and hence is pure. Therefore $S(P|\{N_j^C\}) = 0$. Hence, the expression for $\mathcal{D}(P|C)$ reduces to $S(P) = S(C)$ in the pure case. Again, by Eq. (3.2), MID equals $2S(C) - S(C) = S(C)$, as does entanglement [2]. ∎

Two simple consequences are that for pure bipartite systems, entanglement captures all of the quantumness, and that QD is symmetric in this case. For mixed states, the situation is of course complicated. One fact, however, is the following result.

Theorem 2 $QD \leq MID$.

Proof. Noting that $\rho_A = \mathrm{Tr}_B(\rho_{AB}) = \mathrm{Tr}_B(\Pi(\rho_{AB}))$, we find $Q(\rho_{CP}) = S_\Pi(P|C) - S(P|C)$, where $S_\Pi(P|C)$ is the conditional entropy evaluated on $\Pi(\rho)$, in view of Eq. (3.2). Comparing this with (3.5) we find that $Q(\rho_{CP}) - \mathcal{D}(P|C) = S_\Pi(P|C) - S(P|\{N_j^C\})$, which is always positive for the following reason. Clearly, $S_\Pi(P|\{N_j^C\}) \geq S(P|\{N_j^C\})$. Now, $S_\Pi(P|C) = S(\sum_{j,k} p(j,k)|j,k\rangle_{PC\,PC}\langle j,k|) - S(\sum_j p(j)|j\rangle_{CC}\langle j|) =$
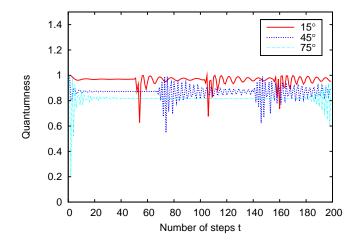

FIG. 3: (Color online) The quantumness using MID and QD, for a unitary walk on a 51-cycle with different coin parameters θ in B_θ, is the same. We note that the frequency of dominant oscillations falls with θ, a behavior we expect from the fact that the speed of a wave packet is proportional to $\sqrt{\cos\theta}$ [33].
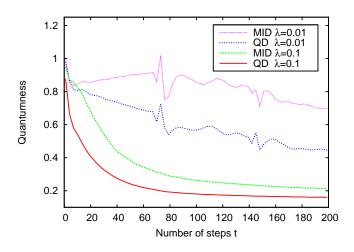

FIG. 4: (Color online) Quantumness on a walk with an increase in noise level on a 51-cycle using $B_{\pi/4}$ as the coin operation. We note that, although both quantumness measures show similar trends including with fluctuations, in the regime of $t \sim 10$ to $t \sim 60$, MID increases (for $\lambda = 0.01$), whereas QD evinces the expected behavior.

$-\sum_{j,k} p(k)p(j|k)\ln(p(j|k)) = \sum_k p(k)S_\Pi(P|C)_k = S_\Pi(P|\{E_j\})$, where $p(j,k)$ is the joint probability of outcomes j and k by measuring $\Pi(\rho_{PC})$ in the eigenbases E_j of their respective reduced density operators, $p(j) \equiv \sum_k p(j,k)$, $p(j|k) = p(j,k)/p(k)$ and $S_\Pi(P|\{E_j\})$ is the average uncertainty in the first register by measuring the second register in the diagonal basis of the latter's density operator. Clearly, $S_\Pi(P|\{E_j\}) \geq S_\Pi(P|\{N_j^C\})$, and we have the required result. ∎

Figure 1 depicts QD and MID for a unitary walk for pure states, which are identical in this case as noted in Theorem 1. We note that whereas the quantumness for

a walk on a line stabilizes eventually, that for a walk on a cycle shows a periodic increase in quantumness, which is associated with 'crossovers', where the left- and right-moving partial waves interfere. Figure 2 shows the expected decrease of quantumness with noise, for both linear and cyclic walks. While MID is seen to upper-bound QD everywhere (except at extremal points, where they are identical), it still tends to reproduce the features of the latter's plot, such as the steep fall and the plateau thereafter.

Figure 3 depicts the θ-dependence of the periodicity of the crossovers of the left- and right- moving components of the walk. This may be understood in terms of the wave-packet dynamics implied by the walk. In Ref. [33], it was shown that the wave velocity obtained by recasting a DTQW as a relativistic-like equation is proportional to $\sqrt{\cos\theta}$.

Figure 4 presents MID and QD as a function of time for two different noise levels. They present a similar degree of sensitivity (with fluctuations roughly in tune with magnitude) and an expected overall reduction with noise. However, MID shows a *rise* in the regime $t \sim 10$ to $t \sim 60$, for the noise parameter $\lambda = 0.01$, which would clearly be unphysical for an indicator of non-classicality, as corroborated by the monotonic fall of QD in this regime. This pathological behavior can be attributed to the non-optimization over local measurements in MID. It may be predicted that if the optimization were performed, the resulting ameliorated MID [7] would show monotonically decreasing behavior. If one could analytically isolate the class of states for which MID applied to a DTQW shows such pathological behavior, and we are able to confirm that the specific instance of walk dynamics does not involve such states, then presumably one could still employ MID as a useful and easy-to-compute indicator of quantumness [26].

IV. CONCLUSION

Noisy quantum walks have been studied from the perspective of comparing MID and QD as indicators of non-classicality, when applied to linear and cyclic DTQWs. MID acts as a loose upper bound to QD, sometimes properly reflecting even fine trends in the latter's behavior. However, there are regimes where it obviously manifests artifacts due to the lack of optimization over local measurements.

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