An environment-mediated quantum deleter

R. Srikanth[1, 2, *] and Subhashish Banerjee[2, †]

[1]*Poornaprajna Institute of Scientific Research, Devanahalli, Bangalore-562 110, India.*
[2]*Raman Research Institute, Sadashiva Nagar, Bangalore- 560 080, India.*

Environment-induced decoherence presents a great challenge to realizing a quantum computer. We point out the somewhat surprising fact that decoherence can be useful, indeed necessary, for practical quantum computation, in particular, for the effective erasure of quantum memory in order to initialize the state of the quantum computer. The essential point behind the deleter is that the environment, by means of a dissipative interaction, furnishes a contractive map towards a pure state. We present a specific example of an amplitude damping channel provided by a two-level system's interaction with its environment in the weak Born-Markov approximation. This is contrasted with a purely dephasing, non-dissipative channel provided by a two-level system's interaction with its environment by means of a quantum nondemolition interaction. We point out that currently used state preparation techniques, for example using optical pumping, essentially perform as quantum deleters.

Quantum computation is well known to solve certain types of problems more efficiently than classical computation [1]. A basic challenge facing the realization of a quantum computer is that of fighting decoherence. Although quantum mechanical linearity endows a quantum computer with greater-than-classical power [2], it also imposes certain restrictions, such as the prohibition on perfect cloning [3] and on deleting the copy of an arbitrary quantum state perfectly [4].

A quantum computational task can be broadly divided into three stages: (1) initializing the quantum computer, by preparing all qubits in a standard 'blank state'; (2) executing the unitary operation that performs the actual computation; (3) performing measurements to read off results.

The well known difficulty in realizing a quantum computer is that of shielding it from the environment during step (2) [5]. This is the problem of fighting decoherence, the loss of coherence in a system (here a qubit), due to interaction with its environment, which has been the subject of intense research. A variety of techniques, including quantum error avoidance [6], quantum error correction [7], quantum error prevention [8], dynamic decoupling [9], frequency modulation of the heat bath [10], fault tolerant quantum computation [11], decoherence-free subspaces [12], among others, exist to combat decoherence. Our main aim here, however, is to point out that in one of the stages of quatum computation, decoherence is useful, even necessary.

In step (1), we must be able to erase quantum memory of the state inherited from a previous computational task, in order to prepare the state of a quantum computer for a subsequent task. What is required is a quantum mechanism that with high probability allows us to pre-pare standard 'blank states', usually designated by the pure state $|0\rangle$. It is clear that no unitary process can achieve this, since true deletion would be irreversible, and hence non-unitary. Given two distinct states $|\psi_1\rangle$ and $|\psi_2\rangle$, a purported deleting operation δ should effect $\delta|\psi_1\rangle \longrightarrow |0\rangle$ and $\delta|\psi_2\rangle \longrightarrow |0\rangle$ [13]. Unitarity requires that $\langle\psi_1|\psi_2\rangle = \langle 0|0\rangle = 1$, which cannot be satisfied unless $\psi_1 = \psi_2$. Further, the no-deleting theorem implies that no qubit state can be erased against a copy [4].

A direct method for initializing the quantum computer would be to measure all qubits in the computational basis. This results in a statistical mixture of $|0\rangle$'s and $|1\rangle$'s, but there is no unitary way in a closed system to flip the $|1\rangle$'s while retaining the $|0\rangle$'s. However, open quantum systems can effect non-unitary evolution on a sub-system of interest. We are thus led to conclude that decoherence is in fact *necessary* for step 1. In particular, decoherence must furnish a contractive map that drives any initial state of the system towards a fixed pure state to serve as the blank state.

In this work, we use this insight to argue that decoherence can be useful to quantum computation. In particular, we show that a dissipative interaction with an environment can serve as an effective deleter of quantum information. Our result is comparable to that presented in Ref. [14], where the evolution of a qubit-probe system is modified by interaction with an environment, which enables the qubit to be driven to a target state, though in function, their work aims for quantum control, whereas the present proposal is aimed at state preparation.

We briefly introduce the interaction of a two-level system (a qubit) with an environment (bath) of harmonic oscillators via a dissipative as well as non-dissipative interaction. Our model assumes that the qubits of the computer are mutually non-interacting and decohere independently. However, it may be noted that in many situations this is a reasonable assumption. In fact, conventional quantum error-correction codes [7] were developed primarily to deal with independent, incoherent errors. In

*Electronic address: srik@rri.res.in
†Electronic address: subhashishb@rri.res.in

the dissipative case the system-environment (S-R) interaction is treated in a standard Born-Markov approximation. In the non-dissipative case, the S-R interaction is of a quantum nondemolition (QND) type. These environmental interactions are representative of open system effects, and further, correspond to two of the most important noisy channels in quantum information theory. In particular, the dissipative type of system-environment interaction yields the (generalized) amplitude damping channel, while the non-dissipative interaction yields the phase damping channel [1]. We point out that whereas the former is useful to engineer a quantum deleter, the latter leads to a mixed state, which is unsuitable for state preparation. This observation is compatible with the fact that erasing information is an irreversible process that dissipates energy [15, 16], affirming the connection between thermodynamics and information theory [17].

The total Hamiltonian is $H = H_S + H_R + H_{SR}$, where H_S, H_R and H_{SR} stand for the Hamiltonians of the system, reservoir and S-R interaction, respectively. Here the system Hamiltonian is given by $H_S = (\hbar\omega/2)\sigma_3$, with σ_3 being the usual Pauli matrix. For the reservoir Hamiltonian we use the standard form of a bath of harmonic oscillators, i.e., $H_R = \hbar\omega_k b_k^\dagger b_k$. We assume separable initial conditions, i.e., $\rho(0) = \rho^s(0)\rho_R(0)$, and the reservoir is assumed to be initially in a squeezed thermal state, i.e., a squeezed thermal bath, with an initial density matrix $\rho_R(0)$ given by

$$\rho_R(0) = S(r,\Phi)\rho_{th}S^\dagger(r,\Phi), \tag{1}$$

where

$$\rho_{th} = \prod_k \left[1 - e^{-\beta\hbar\omega_k}\right]\exp\left(-\beta\hbar\omega_k b_k^\dagger b_k\right) \tag{2}$$

is the density matrix of the thermal bath, and

$$S(r_k, \Phi_k) = \exp\left[r_k\left(\frac{b_k^2}{2}e^{-i2\Phi_k} - \frac{b_k^{\dagger 2}}{2}e^{i2\Phi_k}\right)\right] \tag{3}$$

is the squeezing operator with r_k, Φ_k being the squeezing parameters [18]. Squeezing of the bath has been shown to be useful in the suppression of decay of quantum coherence [19], and to modify the evolution of the geometric phase of two-level quantum systems [21]. Hence it is of relevance to study its possible influence on the behavior of the quantum deleter.

The system-environment interaction is taken to be dissipative and of the weak Born-Markov type [20] leading to a standard Lindblad equation, which in the interaction picture has the following form [21]

$$\frac{d}{dt}\rho^s(t) = \sum_{j=1}^2 \left(2R_j\rho^s R_j^\dagger - R_j^\dagger R_j\rho^s - \rho^s R_j^\dagger R_j\right), \tag{4}$$

where $R_1 = (\gamma_0(N_{th}+1)/2)^{1/2}R$, $R_2 = (\gamma_0 N_{th}/2)^{1/2}R^\dagger$ and $N_{th} = (\exp(\hbar\omega/k_B T) - 1)^{-1}$, is the Planck distribution giving the number of thermal photons at the frequency ω. Here $R = \sigma_- \cosh(r) + e^{i\Phi}\sigma_+ \sinh(r)$, and

the quantities r and Φ are the environmental squeezing parameters and $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$. If $T = 0$, so that $N_{th} = 0$, then R_2 vanishes, and a single Lindblad operator suffices.

From this, the Bloch vectors can be obtained as [21]

$$\langle\sigma_1(t)\rangle = \left[1 + \frac{1}{2}\left(e^{\gamma_0 at} - 1\right)(1 + \cos(\Phi))\right]\beta(t)\langle\sigma_1(0)\rangle$$
$$- \sin(\Phi)\sinh(\frac{\gamma_0 at}{2})e^{-\frac{\gamma_0}{2}(2N+1)t}\langle\sigma_2(0)\rangle,$$

$$\langle\sigma_2(t)\rangle = \left[1 + \frac{1}{2}\left(e^{\gamma_0 at} - 1\right)(1 - \cos(\Phi))\right]\beta(t)\langle\sigma_2(0)\rangle$$
$$- \sin(\Phi)\sinh(\frac{\gamma_0 at}{2})e^{-\frac{\gamma_0}{2}(2N+1)t}\langle\sigma_1(0)\rangle,$$

$$\langle\sigma_3(t)\rangle = e^{-\gamma_0(2N+1)t}\langle\sigma_3(0)\rangle - \frac{(1 - e^{-\gamma_0(2N+1)t})}{2N+1}, \tag{5}$$

where $\beta(t) \equiv \exp(-\frac{\gamma_0}{2}(2N+1+a)t)$, $a = \sinh(2r)(2N_{th}+1)$, $N = N_{th}(\cosh^2(r) + \sinh^2(r)) + \sinh^2(r)$, and γ_0 is a constant typically denoting the system-environment coupling strength, while ω is the system frequency. In Eqs. (5), the quantities $\langle\sigma_j(0)\rangle$ $(j = 1, 2, 3)$ are the expectation values of the respective Pauli operators with respect to the initial state

$$|\psi(0)\rangle = \cos\left(\frac{\theta_0}{2}\right)|1\rangle + e^{i\phi_0}\sin\left(\frac{\theta_0}{2}\right)|0\rangle. \tag{6}$$

It is seen from Eqs. (5) that for fixed squeezing, the asymptotic equilibrium state ρ_{asymp}, given by

$$\rho_{asymp} \equiv \begin{pmatrix} 1-p & 0 \\ 0 & p \end{pmatrix}, \tag{7}$$

where $p = \frac{1}{2}\left[1 + \frac{1}{2N+1}\right]$, is approached faster for stronger coupling, as might be expected. The entire Bloch sphere shrinks towards ρ_{asymp}. In the absence of squeezing and at zero temperature (T), this action corresponds exactly to an amplitude damping channel, for which $N = 0$ and thus $p = 1$. This corresponds to the point representing the state $|0\rangle$ (the Bloch sphere south pole in our notation). For the case of finite T but zero squeezing, this corresponds to a generalized amplitude damping channel, for which $N > 0$, giving $p < 1$, i.e., a mixed state [21]. In the limit of infinite temperature, $N \mapsto \infty$ and $p \mapsto \frac{1}{2}$, so that ρ_{asymp} tends to the maximally mixed state, represented by the center of the Bloch sphere. Thus, the interaction with the environment provides a contractive map, such that the asymptotic state is pure $(p = 1)$ or mixed $(p < 1)$, depending on environmental conditions.

It is worth stressing that the quantum deleter requires a dissipative interaction with its environment. This may be contrasted with a nondissipative interaction, in order to shed light on why the former is necessary for our purpose. In the case of a non-dissipative, quantum nondemolition (QND) interaction, the environment acts as a purely quantum dephasing channel that leaves the energy
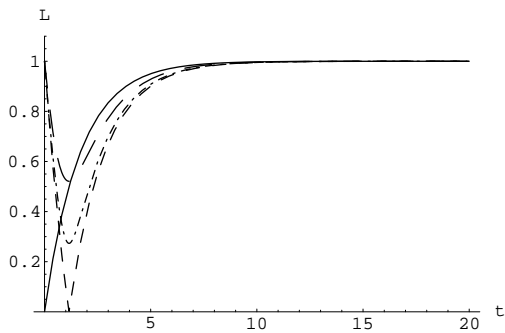
FIG. 1: Evolution of the length L of the Bloch vector (Eqs. (5)) for various initial states, with $\gamma_0 = 0.5$, $\omega = 1.0$, temperature $T = 0$ and the squeezing parameters set to zero. The small-dashed, dot-dashed and large-dashed curves represent initial pure states with $\theta_0 = 0$, $\pi/8$ and $\pi/4$, respectively (in this case, there is no dependence of L on azimuthal angle ϕ_0). The solid curve represents the intially maximally mixed state. In the case of the pure states, note that after at first becoming mixed on account of entanglement with the environment, the qubit returns to purity (in the state $|0\rangle$).

of the system unchanged [21]. This is illustrated briefly in Appendix A.

Fig. 1 depicts the length of the Bloch vector from Eqs. (5) for various initial states (both pure as well as maximally mixed) as a function of time. It can be shown from Eqs. (5) that for all pure states except $|0\rangle$, the quantity $\mathbf{r} \cdot d\mathbf{r}/dt$, where $\mathbf{r}$ is the Bloch vector, is negative at $t = 0$, implying that the tip of the Bloch vector plunges initially into the Bloch sphere on its way towards the point representing the stationary state $|0\rangle$. This is reflected in the initial reduction of the length of the Bloch vector, during which time, the qubit becomes increasingly entangled with the environment, before gradually factoring out.

The proposed quantum deleter works as follows. To clear the memory of the quantum computer, the control processes brought into play to shield it from environmental decoherence, are turned off for a short time t, during which each qubit in the quantum computer is assumed to interact independently with and equilibrate with its environment, assumed to be an unsqueezed vacuum bath. We may characterize the performance of the quantum deleter in terms of fidelity, a measure of closeness of two quantum states [1]. From Eqs. (5), we find that with probability exponentially approaching unity, the fidelity $f(t) \equiv F(\rho^s(t), |0\rangle)$ of a qubit approaches the designated standard 'blank state' $|0\rangle$ according to

$$f(t) = \sqrt{\langle 0|\rho^s(t)|0\rangle} = \sqrt{\frac{1 - \langle \sigma_3(t) \rangle}{2}}$$
$$= \frac{1}{\sqrt{2}} \left[\left(1 - e^{-\Gamma t}\langle \sigma_3(0)\rangle\right) + \frac{\left(1 - e^{-\Gamma t}\right)}{2N + 1} \right]^{1/2} \quad (8)$$

where $\Gamma \equiv \gamma_0(2N + 1)$ and $\langle \sigma_3(0) \rangle$ is the expectation value of σ_3 at time $t = 0$.

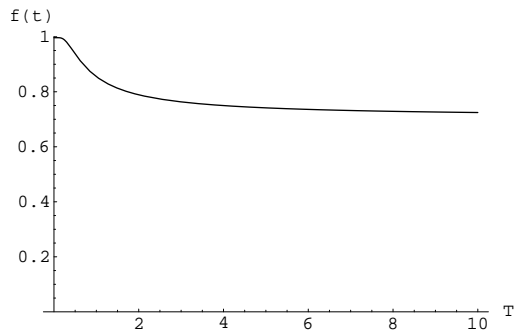Observe that at any time t, fidelity is smallest when

FIG. 2: Fidelity ($f(t)$) falls as a function of temperature (T, in units where $\hbar \equiv k_B \equiv 1$) until it reaches the value $1/\sqrt{2}$ corresponding to a maximally mixed state. The case shown here corresponds to $\theta_0 = 0$, $\gamma_0 = 0.5$, $\omega = 1.0$ and time $t = 10$. Here we set the squeezing parameters r and Φ to zero.
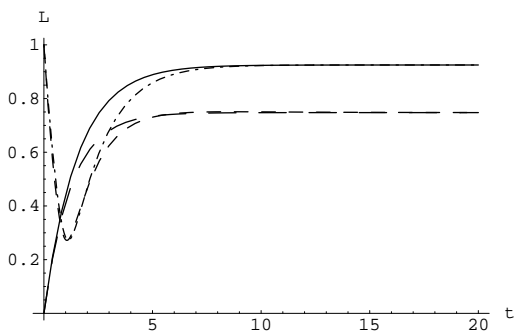

FIG. 3: The effect of squeezing: The bold and the dot-dashed lines correspond to $r = 0.2$, the latter representing an initial pure state with $\theta_0 = \pi/4$, the former an initially maximally mixed state. The large-dashed (maximally mixed initially) and small-dashed (initially pure state with $\theta_0 = \pi/4$) lines represent the evolution of the Bloch vector for the same initial states, but with $r = -0.4$. In all cases, $T = 0$, $\omega = 1.0$ and $\gamma_0 = 0.6$.

$\langle \sigma_3(0) \rangle$ is largest, i.e., 1. Thus, no matter what the initial state, we can lower bound fidelity as

$$1 \geq f(t) \geq \frac{1}{\sqrt{2}} \left[\left(1 - e^{-\Gamma t}\right) \left(1 + \frac{1}{2N + 1}\right) \right]^{1/2}. \quad (9)$$

Observe that if and only if temperature and squeezing are zero, implying $N = 0$, the asymptotic value of $f(t)$ is 1, i.e., $f(\infty) = 1$. If any of these two conditions is not met, then $N > 0$ and hence $f(\infty) < 1$. Therefore, given zero temperature and zero squeezing, one may select the 'off-shield' duration t that guarantees fidelity equal to or higher than any pre-selected value (less than 1).

As seen from Eq. (7), the effect of increasing temperature is to make the asymptotic state ρ_{asymp} ever more mixed. As $T \to \infty$, ρ_{asymp} tends to the maximally mixed state. Fig. 2 depicts the effect of increasing temperature on fidelity. Fig. 3 depict the effect of squeezing on the evolution of the Bloch vector length (from Eqs. (5)). It can be seen that due to the presense of squeezing, the sys-

tem does not return to purity in the state $|0\rangle$. Therefore, for optimal performance of the quantum deleter, i.e., for high purity of the output state of the system qubit, both the squeezing parameter and temperature must be set as close to zero as possible (i.e., $T = 0$ and $r = 0$, yielding $N = 0$ and hence $p = 1$).

It is interesting to note that a contractive map finds another practical, but quite different (cryptographic), use. In the universal quantum homogenizer (UQH), proposed by Ziman *et al.* [22], a sequence of interactions of a system qubit with a reservoir of qubits, initially prepared in an identical state ξ, drives an arbitrary state towards ξ. As a result, the UQH acts as a *quantum safe* with a classical key, consisting of the interaction sequence. One might also consider the UQH being turned into a quantum deleter by setting $\xi \equiv |0\rangle$. However this would require a sequence of highly controlled operations, which would render it relatively difficult to implement, in comparison to allowing a system to cool via interaction with its environment, as required in our case.

We believe that the most interesting aspect of our work is the idea that the unavoidable problem posed by decoherence is shown to be not only useful, but indeed necessary for quantum computation, in particular for initial state preparation. By allowing a role to be played by a realistic environment, in particular, one in which squeezing is absent, it improves the chance that a quantum computer can be practically realized. In practice, what is needed is to turn off in step (1) control processes used to protect the quantum information in step (2) against decoherence due to a vacuum environment. For example, in a quantum information processing system using ultracold atoms in a magneto-optic trap, one may allow spontaneous decay of excitations. However, one should not switch off the trap, since this would effectively bring the system in contact with a finite temperature environment.

It would at first appear that the requirement of a vacuum bath ($T = 0$) places a technological hurdle. However, we note that state preparation techniques in atomic-optical experiments, quantum dots, etc. essentially involve a quantum deletion process, usually furnished by a spontaneous emission. Some examples are, preparation of a Zeeman-state in an optical lattice by two-dimensional sideband Raman cooling [23], and by laser cooling of the spin of an electron trapped in a semiconductor quantum dot [24]. In this context, the notable experiment by Myatt *et al.* [25], in which they engineered a (nearly) zero-temperature reservoir, is worth pointing out.

APPENDIX A: EVOLUTION GOVERNED BY QND S-R INTERACTION

We consider a system interacting with its bath via a QND interaction. The Hamiltonian is

$$H = H_S + \sum_k \hbar\omega_k b_k^\dagger b_k + H_S \sum_k g_k(b_k + b_k^\dagger) + H_S^2 \sum_k \frac{g_k^2}{\hbar\omega_k}. \tag{A1}$$

The second term on the RHS of the above equation is the free Hamiltonian of the environment, while the third term is the S-R interaction Hamiltonian. The last term on the RHS of Eq. (A1) is a renormalization inducing 'counter term'. Since $[H_S, H_{SR}] = 0$, Eq. (A1) is of QND type.

Following Ref. [21], taking into account the effect of the environment modelled as a squeezed thermal bath (Eqs. (1), (2) and (3)), the reduced dynamics of the system can be shown to be

$$\rho_{nm}^s(t) = e^{-\frac{i}{\hbar}(E_n - E_m)t} e^{i(E_n^2 - E_m^2)\eta(t)} e^{-(E_n - E_m)^2 \gamma(t)} \rho_{nm}^s(0), \tag{A2}$$

where the explicit forms of $\eta(t)$ and $\gamma(t)$ can be obtained from Ref. [21]. Here E_n is the eigenstate of the system Hamiltonian defined in the system eigenbasis. It can be seen from the above equation that whereas the off-diagonal elements of the reduced density matrix decay with time, the diagonal elements remain unaffected. Clearly, this feature makes a QND S-R interaction unsuitable for quantum deletion. We note that Eq. (A2) does not depend on the specific form of the system Hamiltonian.

For the case of a two-level system, Eq. (A2) can be recast in the form of Bloch vectors as follows [21]:

$$\begin{aligned}
\langle\sigma_1(t)\rangle &= \sin(\theta_0)\cos(\omega t + \phi_0)e^{-(\hbar\omega)^2\gamma(t)}, \\
\langle\sigma_2(t)\rangle &= \sin(\theta_0)\sin(\omega t + \phi_0)e^{-(\hbar\omega)^2\gamma(t)}, \\
\langle\sigma_3(t)\rangle &= \cos(\theta_0),
\end{aligned} \tag{A3}$$

where $\gamma = \gamma(\gamma_0, T, r, \Phi) \geq 0$ ($\gamma = 0$ if and only if the environmental interaction is absent) is calculated for the $T = 0$ and high T case in Ref. [21]. It suffices for our present purpose to note that the action described by Eqs. (A3) is that of contracting the Bloch sphere along the σ_3-axis. An initial pure state is driven to a mixed state that is diagonal in the computational basis, *even at $T = 0$*. Thus, this is not useful for our purpose.

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